



ROYAL UNIVERSITY OF PHNOM PENH UNIVERSITÉ ROYALE DE PHNOM PENH

A Differential Equation with Grazing Unilateral Contact

Thesis Report Submitted
in Partial Fulfillment of the Requirement for
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1 Introduction

The project aims to study the exact solutions, the asymptotic expansion by the Lindstedt-Poincaré method and a numerical study. It also intends to study the validity of the method of strained coordinates with the very critical example of the spring motion modeled by the nonlinear differential equations with grazing unilateral contact and one degree of freedom. The nonlinearity of the spring motion is modeled by the equation $u''(t) + u(t) + \epsilon[u(t) - 1]_+ = 0$ with the function $(u)_+ = \max(u, 0)$.

For the spring model, the structure has an unilateral spring of rigidity ϵ and that one spring is classical linear and is attached to the mass and to a rigid wall, the second spring is still linear and is attached to a rigid wall but has an unilateral contact to the mass. (See Figure 1 below). A critical example is given

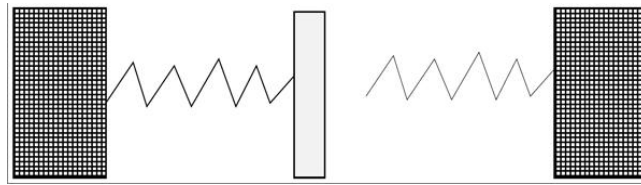


Figure 1

by the following system:

$$\begin{cases} u''(t) + u(t) + \epsilon[u(t) - 1]_+ = 0 \\ u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases} \quad (1)$$

We will study the exact solutions for cases $\epsilon = 0$, $-1 < \epsilon < 0$ and $0 < \epsilon \leq 1$. The interesting and critical case is $0 < \epsilon \leq 1$ while the other cases are only the introductory cases of our purpose. We will also compare the exact solution for the case $\epsilon > 0$ to the solution given by the Lindstedt-Poincaré method: namely the solution of the form $u(t) = (1 + \epsilon) \cos(t)$ and see if the exact solution and the Lindstedt-Poincaré solution are closed on the intervals $[0, 1]$, $[0, \frac{1}{\epsilon}]$ and $[0, \frac{1}{\sqrt{\epsilon}}]$.

2 Exact Solutions

2.1 Case $\epsilon = 0$

For this case, the equation (1) becomes:

$$\begin{cases} u''(t) + u(t) = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases} \quad (2)$$

The characteristic polynomial for $u''(t) + u(t) = 0$ is $r^2 + 1 = 0$. The solution of the equation (2) is determined by the roots of this polynomial. The polynomial has the roots $r = \pm i$ where $i^2 = -1$. Thus the general solution is:

$$u(t) = A \cos(t) + B \sin(t), \text{ where } A, B \text{ are constants}$$

We also have $u'(t) = -A \sin(t) + B \cos(t)$, so we apply the initial conditions $\begin{cases} u(0) = 1 \\ u'(0) = 0 \end{cases}$, then we get the exact solution for $\epsilon = 0$:

$$\boxed{u(t) = \cos(t)}$$

2.2 Case $-1 < \epsilon < 0$

We have $u(0) = 1 + \epsilon$, then $u(0) < 1$. Assume that $u(t) < 1$ for all t then we will find the solution for this case. We will also show that our assumption stays valid that is $u(t) < 1$ for all t .

- If $u(t) < 1$ for all t

then we have the equation without nonlinear term $(u - 1)_+$, so the equation (1) becomes:

$$\begin{cases} u''(t) + u(t) = 0 \\ u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases} \quad (3)$$

In similar way, the equation $u''(t) + u(t) = 0$ has the general solution

$$u(t) = C \sin(t) + D \cos(t), \quad \text{where } C \text{ and } D \text{ are constants.}$$

With $u'(t) = C \cos(t) - D \sin(t)$, we apply the initial conditions $\begin{cases} u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases}$ then we get the exact solution:

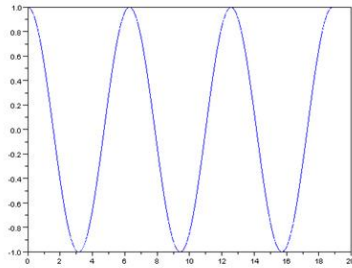
$$\boxed{u(t) = (1 + \epsilon) \cos(t)}$$

We can verify easily that the solution stays less than 1.

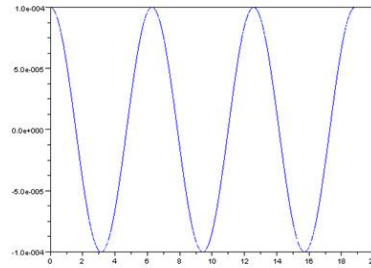
We have $u(0) = 1 + \epsilon$, then $u(0) < 1$. Bound the exact solution for $t \neq 0$, we get

$$|u(t)| = |(1 + \epsilon) \cos(t)| = |1 + \epsilon| |\cos(t)|$$

Since $-1 < \epsilon < 0 \implies |1 + \epsilon| < 1$, and $|\cos(t)| \leq 1$, for all t then we get $|u(t)| < 1$ for $t \neq 0$. Consequently, $u(t) < 1$ for all t . You can see from the plot below that the solution is less than 1.



$$\epsilon = -0.0001$$



$$\epsilon = -0.9999$$

We can also verify this result with the energy method that $u(t) < 1$ for all t . Firstly, consider the energy for the general problem $u''(t) + u(t) + \epsilon[u(t) - 1]_+ = 0$. Multiply this equation by u' yields

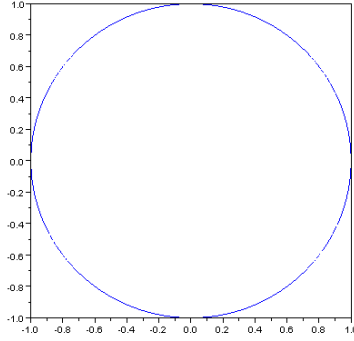
$$u''u' + uu' + \epsilon u'[u - 1]_+ = 0 \quad (4)$$

Integrate (4) between 0 and t with:

$$\begin{cases} \frac{d}{2dt} u'^2 = u'' u' \\ \frac{d}{2dt} u^2 = u u' \\ \frac{d}{2dt} [(u-1)_+]^2 = u'(u-1)_+ \end{cases}$$

yields the energy $E(u, u') = u'^2 + u^2 + \epsilon[(u-1)_+]^2$. We see that with $\epsilon = 0$, we have the energy for the linear problem $u'' + u = 0$ that is $E(u, u') = u'^2 + u^2$, and it is constant since $\frac{d}{dt} E(u, u') = 0$. With initial conditions $\begin{cases} u(0) = 1 \\ u'(0) = 0 \end{cases}$ at time $t = 0$, we have $E_0 = 1$, so the parametric curves $(u(t), u'(t))$ are on the level set of phase space $E(u, u') = E_0$ defined by the circle $u'^2 + u^2 = 1$.

In similar way for $-1 < \epsilon < 0$ and at the initial time with the conditions $\begin{cases} u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases}$, we have $E_0 = (1 + \epsilon)^2$. Still, the energy is constant and the parametric curves $(u(t), u'(t))$ lie on the phase space $E(u, u') = E_0$, that is the space defined by the circle $u'^2(t) + u^2(t) = (1 + \epsilon)^2$ with radius less than 1, so the solution is periodic. Since the parametric curves $(u(t), u'(t))$ lie on the circle $E(u, u') = E_0$ and $E_0 = (1 + \epsilon)^2 < 1$, then $u < 1$ which implies that $u(t) < 1$ for all t . The solution $u(t) = (1 + \epsilon) \cos(t) < 1$ for $\epsilon < 0$ and lies on the level set of phase space defined by the circle. The plot of the phase space with $\epsilon = -0.0001$ illustrates that the maximum solution stays less than 1.



Phase Space $u'^2 + u^2 = (1 + \epsilon)^2$

2.3 Case $0 < \epsilon \leq 1$

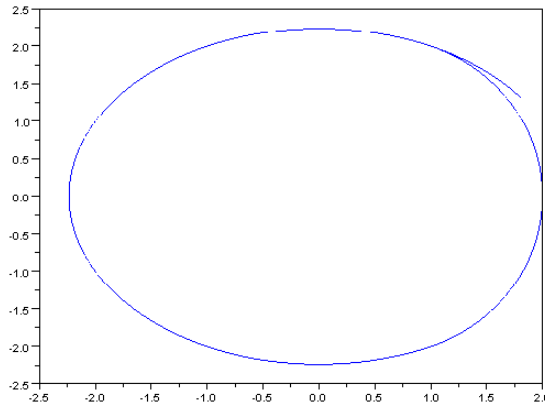
From previous case, the energy of the problem with nonlinear term $(u-1)_+$, $u''(t) + u(t) + \epsilon[u(t) - 1]_+ = 0$, is $E(u, u') = u'^2 + u^2 + \epsilon[(u-1)_+]^2$. At the initial time $t = 0$ and we use the initial conditions

$$\begin{cases} u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases}, \quad 0 < \epsilon \leq 1$$

we get

$$E_0 = 1 + 2\epsilon + \epsilon^2 + \epsilon[(\epsilon)_+]^2 = \epsilon^3 + \epsilon^2 + 2\epsilon + 1$$

So, the parametric curves are in the level set of phase space $E(u, u') = E_0$ defined by the ellipse $u'^2 + u^2 + \epsilon[(u-1)_+]^2 = \epsilon^3 + \epsilon^2 + 2\epsilon + 1$. The phase space defined by two pieces of ellipses as in the following figure.



We have $u(t) = 1 + \epsilon > 1$ for $t = 0$ and since the parametric curves $(u(t), u'(t))$ lie in the level set of phase space then there exists t_1 such that for $[0, t_1[$ we have $u(t) > 1$, and there exists t_2 such that for $]t_1, t_2[$ we have $u(t) < 1$.

- If $0 \leq t < t_1 \implies u(t) > 1$

then we have the problem with the nonlinear term $(u - 1)_+ = u - 1$. So, the problem for this case is

$$\begin{cases} u''(t) + (1 + \epsilon)u(t) = \epsilon \\ u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases} \quad (\text{IVP1})$$

Using the method of variation of parameters, we see that the exact solution of the IVP1 above is of the form $u(t) = u_h(t) + u_p(t)$ where $u_h(t)$ is the solution of the homogeneous equation $u''(t) + (1 + \epsilon)u(t) = 0$ and $u_p(t)$ is the particular solution of the non-homogeneous equation $u''(t) + (1 + \epsilon)u(t) = \epsilon$ obtained from the variation of parameters.

The homogeneous equation $u''(t) + (1 + \epsilon)u(t) = 0$ has the corresponding characteristic polynomial $r^2 + (1 + \epsilon) = 0$. Since the roots of this polynomial are $r = \pm\sqrt{1 + \epsilon}i$, then the homogeneous solution is

$$u_h(t) = C_1 \cos(\sqrt{1 + \epsilon}t) + C_2 \sin(\sqrt{1 + \epsilon}t) \text{ where } C_1 \text{ and } C_2 \text{ are constants.}$$

We vary the term C_1 and C_2 as $u_1(t)$ and $u_2(t)$, so the particular solution for non-homogeneous equation is of the form

$$u_p(t) = u_1(t) \cos(\sqrt{1 + \epsilon}t) + u_2(t) \sin(\sqrt{1 + \epsilon}t)$$

where $u_1(t)$ and $u_2(t)$ satisfy

$$\begin{cases} u_1'(t) \cos(\sqrt{1 + \epsilon}t) + u_2'(t) \sin(\sqrt{1 + \epsilon}t) = 0 \\ -u_1'(t)\sqrt{1 + \epsilon} \sin(\sqrt{1 + \epsilon}t) + u_2'(t)\sqrt{1 + \epsilon} \cos(\sqrt{1 + \epsilon}t) = \epsilon \end{cases}$$

Solve this system we get

$$\begin{cases} u_1'(t) = -\frac{\epsilon}{\sqrt{1 + \epsilon}} \sin(\sqrt{1 + \epsilon}t) \\ u_2'(t) = \frac{\epsilon}{\sqrt{1 + \epsilon}} \cos(\sqrt{1 + \epsilon}t) \end{cases}$$

By integration, $u_1(t)$ and $u_2(t)$ are of the forms:

$$\begin{cases} u_1(t) = \frac{\epsilon}{1+\epsilon} \cos(\sqrt{1+\epsilon t}) \\ u_2(t) = \frac{\epsilon}{1+\epsilon} \sin(\sqrt{1+\epsilon t}) \end{cases}$$

Plug this latter results into the form of our general solution, we obtain

$$u(t) = C_1 \cos(\sqrt{1+\epsilon t}) + C_2 \sin(\sqrt{1+\epsilon t}) + \frac{\epsilon}{1+\epsilon} [\cos^2(\sqrt{1+\epsilon t}) + \sin^2(\sqrt{1+\epsilon t})]$$

then we get

$$u(t) = \frac{\epsilon}{1+\epsilon} + C_1 \cos(\sqrt{1+\epsilon t}) + C_2 \sin(\sqrt{1+\epsilon t}), \text{ for all } t.$$

With $u'(t) = -C_1 \sqrt{1+\epsilon} \sin(\sqrt{1+\epsilon t}) + C_2 \sqrt{1+\epsilon} \cos(\sqrt{1+\epsilon t})$, we can apply the initial conditions $\begin{cases} u(0) = 1 + \epsilon \\ u'(0) = 0 \end{cases}$ to find the values of C_1 and C_2 . Then we get $C_1 = \frac{\epsilon^2 + \epsilon + 1}{1 + \epsilon}$ and $C_2 = 0$. Thus we have

$$\begin{aligned} u(t) &= \frac{\epsilon}{1+\epsilon} + \frac{\epsilon^2 + \epsilon + 1}{1+\epsilon} \cos(\sqrt{1+\epsilon t}) \\ &= \frac{\epsilon}{1+\epsilon} + \left(1 + \frac{\epsilon^2}{1+\epsilon}\right) \cos(\sqrt{1+\epsilon t}) \end{aligned}$$

The exact solution of our IVP1 for $0 \leq t < t_1$ is

$$\boxed{u(t) = \frac{\epsilon}{1+\epsilon} + \left(1 + \frac{\epsilon^2}{1+\epsilon}\right) \cos(\sqrt{1+\epsilon t})}$$

By the symmetric property of the cosine function, this equality is also valid on time interval $] -t_1, 0[$. Since cosine is a decreasing function on $[0, \frac{\pi}{2}]$ from 1 to 0, there exists t_1 such that $u(t_1) = 1$, furthermore, $u'(t_1) < 0$ and t_1 is the unique solution in $]0, \frac{\pi}{2}[$ of the equation

$$\begin{aligned} 1 &= \frac{\epsilon}{1+\epsilon} + \left(1 + \frac{\epsilon^2}{1+\epsilon}\right) \cos(\sqrt{1+\epsilon t_1}) \\ \Leftrightarrow 1 - \frac{\epsilon}{1+\epsilon} &= \frac{1+\epsilon+\epsilon^2}{1+\epsilon} \cos(\sqrt{1+\epsilon t_1}) \\ \Leftrightarrow \cos(\sqrt{1+\epsilon t_1}) &= \frac{1}{1+\epsilon+\epsilon^2} \end{aligned} \quad (*_1)$$

So, we get

$$t_1 = \frac{\arccos\left(\frac{1}{1+\epsilon+\epsilon^2}\right)}{\sqrt{1+\epsilon}}$$

Notice that $0 < t_1 = t_1(\epsilon) < \frac{\pi}{2}$ for $\epsilon > 0$ and $t_1(0) = 0$. Thus, $u(t) > 1$ on $I_1 =] -t_1, 0] \cup [0, t_1[$.

- If $t \in]t_1, t_2[$, $\implies u(t) < 1$

Now the differential equation is much simpler without nonlinear term that is: $u''(t) + u(t) = 0$, but the initial conditions at $t = t_1$ are more complex. At t_1 ,

the behavior of $u(\cdot)$ turns from greater than 1 to less than 1, then it's obvious that $u(t_1) = 1$. With the solution $u(t) = \frac{\epsilon}{1+\epsilon} + (1 + \frac{\epsilon^2}{1+\epsilon}) \cos(\sqrt{1+\epsilon}t)$, we have

$$\begin{aligned} u'(t) &= -\sqrt{1+\epsilon} \left(1 + \frac{\epsilon^2}{1+\epsilon}\right) \sin(\sqrt{1+\epsilon}t) \\ &= -\left(\sqrt{1+\epsilon} + \frac{\epsilon^2}{\sqrt{1+\epsilon}}\right) \sin(\sqrt{1+\epsilon}t) \end{aligned}$$

For $u(t) < 1$, we get the following conditions at initial time $t = t_1$:

$$\begin{cases} u(t_1) = 1 \\ u'(t_1) = -\left(\sqrt{1+\epsilon} + \frac{\epsilon^2}{\sqrt{1+\epsilon}}\right) \sin(\sqrt{1+\epsilon}t_1) \end{cases}$$

We have the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and from $(*_1)$ we can replace

$$\begin{aligned} 0 < \sin(\sqrt{1+\epsilon}t_1) &= \sqrt{1 - \cos^2(\sqrt{1+\epsilon}t_1)} \\ &= \sqrt{1 - \frac{1}{(1+\epsilon+\epsilon^2)^2}} \end{aligned}$$

then we obtain $u'(t_1)$ has only a function of ϵ such that

$$\begin{aligned} u'(t_1) &= -\left(\sqrt{1+\epsilon} + \frac{\epsilon^2}{\sqrt{1+\epsilon}}\right) \sqrt{1 - \frac{1}{(1+\epsilon+\epsilon^2)^2}} \\ &= -\frac{\epsilon^2 + \epsilon + 1}{\sqrt{1+\epsilon}} \sqrt{1 - \frac{1}{(1+\epsilon+\epsilon^2)^2}} \\ &= -\sqrt{\frac{(\epsilon^2 + \epsilon + 1)^2}{1+\epsilon} - \frac{1}{1+\epsilon}} \\ &= -\sqrt{\frac{(\epsilon^2 + \epsilon + 1)^2 - 1}{1+\epsilon}} \\ &= -\sqrt{\frac{(\epsilon^2 + \epsilon)(\epsilon^2 + \epsilon + 2)}{1+\epsilon}} \\ &= -\sqrt{\epsilon(\epsilon^2 + \epsilon + 2)} < 0 \end{aligned}$$

Our problem on $]t_1, t_2[$ becomes:

$$\begin{cases} u''(t) + u(t) = 0 \\ u(t_1) = 1 \\ u'(t_1) = -\sqrt{\epsilon(\epsilon^2 + \epsilon + 2)} \end{cases}, \text{ where } t_1 = \frac{\arccos(\frac{1}{1+\epsilon+\epsilon^2})}{\sqrt{1+\epsilon}} \quad (\text{IVP2})$$

Like the previous case, the general solution for the equation $u''(t) + u(t) = 0$ is $u(t) = A \sin(t) + B \cos(t)$ where A and B are constants.

With $u'(t) = A \cos(t) - B \sin(t)$ and after we apply the initial conditions at $t = t_1$, we get the solution on $]t_1, t_2[$ as:

$$u(t) = \cos(t - t_1) + u'(t_1) \sin(t - t_1)$$

Thus, on $]t_1, t_2[$, the solution for IVP2 is:

$$u(t) = \cos(t - t_1) - \sqrt{\epsilon(\epsilon^2 + \epsilon + 2)} \sin(t - t_1)$$

The number t_2 is the first real greater than t_1 and that the behavior of $u(\cdot)$ turn from less than 1 to greater than 1, so it can be found from the equation $u(t_2) = 1$ below:

$$1 = \cos(t_2 - t_1) + u'(t_1) \sin(t_2 - t_1), \text{ where } t_1 < t_2 < t_1 + 2\pi$$

Using the asymptotic expansion with $t_1 = \frac{\arccos(\frac{1}{1+\epsilon+\epsilon^2})}{\sqrt{1+\epsilon}}$, we can give an approximation of t_2 by using different methods and one of which is the energy method.

2.4 Period and Asymptotic Expansion of the Period

We can compute the period of the solution explicitly from the energy. From previous section, the energy of the problem $u''(t) + u(t) + \epsilon[u(t) - 1]_+ = 0$ is $E(u, u') = u'^2 + u^2 + \epsilon[(u-1)_+]^2 = (u')^2 + F(u)$ where $F(u) = u^2 + \epsilon[(u-1)_+]^2$. We see that

$$F(u) = \begin{cases} u^2 & \text{if } u < 1 \\ u^2 + \epsilon(u-1)^2 & \text{if } u > 1 \end{cases}$$

The physical energy is $\frac{E_0}{2}$.

Let us define p and q , the extreme values of $u(\cdot)$ such that

$$p < 0 < q, \quad F(p) = F(q) = E_0$$

For $0 < \epsilon \leq 1$, we have

$$\begin{aligned} E_0 &= 0^2 + (1 + \epsilon)^2 + \epsilon[(\epsilon)_+]^2 \\ &= \epsilon^3 + \epsilon^2 + 2\epsilon + 1 \end{aligned}$$

Then we define:

$$p = -\sqrt{E_0} = -\sqrt{\epsilon^3 + \epsilon^2 + 2\epsilon + 1} < 0, \text{ and } q = 1 + \epsilon > 0$$

Notice that

$$\begin{aligned} F(p) &= F(-\sqrt{\epsilon^3 + \epsilon^2 + 2\epsilon + 1}) \\ &= (-\sqrt{E_0})^2 + \epsilon\{[-(\sqrt{E_0} + 1)]_+\}^2 \\ &= E_0 + \epsilon\{[-(\sqrt{E_0} + 1)]_+\}^2 \end{aligned}$$

Since

$$\begin{aligned} -(E_0 + 1) &= -(\sqrt{\epsilon^3 + \epsilon^2 + 2\epsilon + 1} + 1) < 0 \\ \implies [-(\sqrt{\epsilon^3 + \epsilon^2 + 2\epsilon + 1} + 1)]_+ &= 0 \\ \implies F(p) &= \epsilon^3 + \epsilon^2 + 2\epsilon + 1 = E_0 \end{aligned}$$

Similarly,

$$\begin{aligned} F(q) &= F(1 + \epsilon) = (1 + \epsilon)^2 + \epsilon[(\epsilon + 1 - 1)_+]^2 \\ &= \epsilon^2 + 2\epsilon + 1 + \epsilon[(\epsilon)_+]^2 \\ \implies F(q) &= \epsilon^3 + \epsilon^2 + 2\epsilon + 1 = E_0 \end{aligned}$$

The period $T(\epsilon)$ is given by:

$$\frac{T(\epsilon)}{2} = \int_p^q \frac{du}{\sqrt{E_0 - F(u)}}$$

Sine $F(u)$ changes when u becomes greater than 1, so, the half period is divided into two parts such that:

$$\frac{T(\epsilon)}{2} = T_1 + T_2$$

where

$$T_1 = \int_p^1 \frac{du}{\sqrt{E_0 - F(u)}} \text{ and } T_2 = \int_1^q \frac{du}{\sqrt{E_0 - F(u)}}$$

- Computation and Expansion of T_1

We have $u \in [p, 1], \implies u < 1$, thus $F(u) = u^2$. then

$$\begin{aligned} T_1 &= \int_{-\sqrt{E_0}}^1 \frac{du}{\sqrt{E_0 - u^2}} = \left[\arcsin\left(\frac{u}{\sqrt{E_0}}\right) \right]_{-\sqrt{E_0}}^1 \\ \implies T_1 &= \arcsin\left(\frac{1}{\sqrt{E_0}}\right) - \arcsin\left(\frac{-\sqrt{E_0}}{\sqrt{E_0}}\right) \\ &= \arcsin(1) + \arcsin\left(\frac{1}{\sqrt{E_0}}\right) \\ \implies T_1 &= \frac{\pi}{2} + \arcsin\left(\frac{1}{\sqrt{1 + \epsilon^3 + \epsilon^2 + 2\epsilon}}\right) \end{aligned}$$

Using the expansion

$$\frac{1}{\sqrt{1+h}} = 1 - \frac{h}{2} + \frac{3}{8}h^2 + O(h^3)$$

we have

$$\begin{aligned} \frac{1}{\sqrt{1 + \epsilon^3 + \epsilon^2 + 2\epsilon}} &= 1 - \frac{\epsilon^3 + \epsilon^2 + 2\epsilon}{2} + \frac{3}{8}(\epsilon^3 + \epsilon^2 + 2\epsilon)^2 + O(\epsilon^3) \\ &= 1 - \epsilon - \frac{\epsilon^2}{2} + \frac{3}{2}\epsilon^2 + O(\epsilon^3) \\ &= 1 - \epsilon + \epsilon^2 + O(\epsilon^3) \end{aligned}$$

Then we have

$$\arcsin\left(\frac{1}{\sqrt{1 + \epsilon^3 + \epsilon^2 + 2\epsilon}}\right) = \arcsin(1 - \epsilon + \epsilon^2)$$

Expand this by using $\arcsin(1+h) = \frac{\pi}{2} - \sqrt{2h} - \frac{h\sqrt{h}}{6\sqrt{2}} + O(h^{2.5})$, then

$$\begin{aligned} \arcsin(1 - \epsilon + \epsilon^2) &= \frac{\pi}{2} - \sqrt{2(\epsilon - \epsilon^2)} - \frac{(\epsilon - \epsilon^2)\sqrt{\epsilon - \epsilon^2}}{6\sqrt{2}} + O(\epsilon^{2.5}) \\ &= \frac{\pi}{2} - \sqrt{2\epsilon}\sqrt{1 - \epsilon} - \frac{(\epsilon^{1.5} - \epsilon^{2.5})\sqrt{1 - \epsilon}}{6\sqrt{2}} + O(\epsilon^{2.5}) \end{aligned}$$

But $\sqrt{1-\epsilon} = 1 - \frac{\epsilon}{2} - \frac{1}{8}\epsilon^2 + O(\epsilon^3)$

$$\begin{aligned}\implies \arcsin(1 - \epsilon + \epsilon^2) &= \frac{\pi}{2} - \sqrt{2\epsilon}\left(1 - \frac{\epsilon}{2} - \frac{1}{8}\epsilon^2\right) - \frac{(\epsilon^{1.5} - \epsilon^{2.5})(1 - \frac{\epsilon}{2} - \frac{1}{8}\epsilon^2)}{6\sqrt{2}} + O(\epsilon^{2.5}) \\ &= \frac{\pi}{2} - \sqrt{2\epsilon} + \frac{5\epsilon^{1.5}}{6\sqrt{2}} + O(\epsilon^{2.5})\end{aligned}$$

We get,

$$\begin{aligned}T_1 &= \frac{\pi}{2} + \arcsin\left(\frac{1}{\sqrt{1 + \epsilon^3 + \epsilon^2 + 2\epsilon}}\right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} - \sqrt{2\epsilon} + \frac{5\epsilon^{1.5}}{6\sqrt{2}} + O(\epsilon^{2.5}) \\ \implies T_1 &= \pi - \sqrt{2\epsilon} + \frac{5\epsilon^{1.5}}{6\sqrt{2}} + O(\epsilon^{2.5})\end{aligned}\quad (*_2)$$

• Computation and Expansion for T_2

We have

$$T_2 = \int_1^q \frac{du}{\sqrt{E_0 - F(u)}} \quad \text{where } q = 1 + \epsilon \text{ and } E_0 = 1 + \epsilon^3 + \epsilon^2 + 2\epsilon$$

We also $u \in [1, q]$, $u > 1$, then $F(u) = (1 + \epsilon)u^2 - 2\epsilon u + \epsilon$

$$\implies T_2 = \int_1^q \frac{du}{\sqrt{E_0 - [(1 + \epsilon)u^2 - 2\epsilon u + \epsilon]}}$$

Using the formula

$$\frac{1}{\sqrt{a}} \arcsin\left(\frac{aU + b}{\sqrt{B}}\right) = \int_{-b/a}^U \frac{dx}{\sqrt{A - (ax^2 + 2bx + c)}}, \quad A, a > 0, B = a(A - c) + b^2 > 0$$

In our case, we have

$$A = E_0, \quad a = 1 + \epsilon = q, \quad b = -\epsilon, \quad c = \epsilon, \quad \text{and } B = (1 + \epsilon)(E_0 - \epsilon) + \epsilon^2$$

Then we get

$$T_2 = \frac{1}{\sqrt{1 + \epsilon}} \left[\arcsin\left(\frac{aq + b}{\sqrt{B}}\right) - \arcsin\left(\frac{a + b}{\sqrt{B}}\right) \right]$$

where

$$\frac{1}{\sqrt{1 + \epsilon}} = 1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 + O(\epsilon^3) \quad (5)$$

$$\begin{aligned}B &= (1 + \epsilon)(1 + \epsilon + \epsilon^2 + \epsilon^3) + \epsilon^2 \\ &= 1 + 2\epsilon + 3\epsilon^2 + 2\epsilon^3 + O(\epsilon^4) \\ \implies \frac{1}{\sqrt{B}} &= \frac{1}{\sqrt{1 + 2\epsilon + 3\epsilon^2 + 2\epsilon^3}} \\ &= 1 - \epsilon - \frac{3}{2}\epsilon - \epsilon^3 + \frac{3}{8}(4\epsilon^2 + 12\epsilon^3) - \frac{5}{2}\epsilon^3 + O(\epsilon^4) \\ &= 1 - \epsilon - \frac{5}{4}\epsilon^3 + O(\epsilon^4)\end{aligned}$$

$$\frac{a+b}{\sqrt{B}} = \frac{1+\epsilon-\epsilon}{\sqrt{B}} = \frac{1}{\sqrt{B}} = 1-\epsilon - \frac{5}{4}\epsilon^3 + O(\epsilon^4) = 1-\epsilon + O(\epsilon^3) \quad (6)$$

$$\begin{aligned} \frac{aq+b}{\sqrt{B}} &= \frac{(1+\epsilon)^2-\epsilon}{\sqrt{B}} = \frac{1+\epsilon+\epsilon^2}{\sqrt{B}} = (1+\epsilon+\epsilon^2)[1-\epsilon - \frac{5}{4}\epsilon^3 + O(\epsilon^4)] \\ &\implies \frac{aq+b}{\sqrt{B}} = 1 - \frac{9}{4}\epsilon^3 + O(\epsilon^4) \end{aligned} \quad (7)$$

From (5), (6), (7) we get

$$\begin{aligned} \arcsin\left(\frac{a+b}{\sqrt{B}}\right) &= \arcsin[1-\epsilon + O(\epsilon^3)] \\ &= \frac{\pi}{2} - \sqrt{2}\epsilon + \frac{5\epsilon^{1.5}}{6\sqrt{2}} + O(\epsilon^{2.5}) \\ \arcsin\left(\frac{aq+b}{\sqrt{B}}\right) &= \arcsin\left(1 - \frac{9}{4}\epsilon^3 + O(\epsilon^4)\right) \\ &= \frac{\pi}{2} - \frac{3}{\sqrt{2}}\epsilon^{1.5} + O(\epsilon^{2.5}) \\ \implies \arcsin\left(\frac{aq+b}{\sqrt{B}}\right) - \arcsin\left(\frac{a+b}{\sqrt{B}}\right) &= \left[\frac{\pi}{2} - \frac{3}{\sqrt{2}}\epsilon^{1.5}\right] - \left[\frac{\pi}{2} - \sqrt{2}\epsilon + \frac{5\epsilon^{1.5}}{6\sqrt{2}}\right] + O(\epsilon^{2.5}) \\ \implies \arcsin\left(\frac{aq+b}{\sqrt{B}}\right) - \arcsin\left(\frac{a+b}{\sqrt{B}}\right) &= \sqrt{2}\epsilon - \frac{23}{6\sqrt{2}}\epsilon^{1.5} + O(\epsilon^{2.5}) \end{aligned}$$

Then we have

$$\begin{aligned} T_2 &= \left[1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 + O(\epsilon^3)\right]\left[\sqrt{2}\epsilon - \frac{23}{6\sqrt{2}}\epsilon^{1.5} + O(\epsilon^{2.5})\right] \\ &= \sqrt{2}\epsilon - \frac{29}{6\sqrt{2}}\epsilon^{1.5} + O(\epsilon^{2.5}) \end{aligned} \quad (*_3)$$

Using (*₂) and (*₃), we get

$$\begin{aligned} \frac{T(\epsilon)}{2} = T_1 + T_2 &= \left[\pi - \sqrt{2}\epsilon + \frac{5\epsilon^{1.5}}{6\sqrt{2}}\right] + \left[\sqrt{2}\epsilon - \frac{29}{6\sqrt{2}}\epsilon^{1.5}\right] + O(\epsilon^{2.5}) \\ &= \pi - 2\sqrt{2}\epsilon^{1.5} + O(\epsilon^{2.5}) \end{aligned}$$

Thus,

$$\boxed{T(\epsilon) = 2\pi - 4\sqrt{2}\epsilon^{1.5} + O(\epsilon^{2.5})}$$

2.5 The Precision of the Period

In this section, we will consider the precision of $T(\epsilon)$ of our problem. In general, if $f(\cdot)$ is a smooth function then we can find the solution $u(\cdot)$ for an equation

$$u'' + u + \epsilon f(u) = 0 \quad (8)$$

with prescribed initial data. It is known that for small ϵ the solution $u(\cdot)$ of (8) is periodic, and the period admits an expansion $T = T(\epsilon) = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots$

Similarly, the pulsation admits an expansion $\omega = \omega(\epsilon) = \frac{2\pi}{T(\epsilon)} = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots$. The solution of (8) then can be written as the following form:

$$u(t) = v[\omega(\epsilon)t]$$

where $v(\cdot)$ is a periodic function with period 1. Assume that $v(\cdot)$ is known everywhere then if we know only T_0 , the first approximation of T , we will know $\omega_0 = \frac{2\pi}{T_0}$. Then $u(t) = v(\omega_0 t) + O(\epsilon)$ on fixed interval $[0, a]$. But on bigger interval $[0, \frac{a}{\epsilon}]$, the error can be non small since

$$u\left(\frac{a}{\epsilon}\right) = v\left(\omega_0 \frac{a}{\epsilon} + \omega_1 a + \omega_2 a \epsilon + \dots\right) = v\left(\omega_0 \frac{a}{\epsilon} + \omega_1 a\right) \neq v\left(\omega_0 \frac{a}{\epsilon}\right)$$

If we want to have an approximation of the first order on $[0, \frac{a}{\epsilon}]$, it suffices to know an approximation of the period of the second order i.e. $T_0 + \epsilon T_1 = T + O(\epsilon^2)$. In practice, with Lindstedt-Poincaré method, $v(\cdot)$ is only known at the first order, i.e. the method gives a one periodic function $w(\cdot)$ such that $w(\cdot) = v(\cdot) + O(\epsilon)$ on $[0, 1]$. It is easy to see that the previous discussion stays true.

In a simplified model of unilateral contact of our problem with the equation $u'' + u + \epsilon f(u) = 0$ involves a non perfectly smooth function $f(u) = (u - 1)_+ = \max(0, u - 1)$. We see that the solution is still periodic for small ϵ , but the period does not admit the same regular expansion. With the initial conditions $u(0) = 1 + \epsilon$, $u'(0) = 0$, it can be proved after tedious computations that the period $T(\epsilon) = 2\pi - c\epsilon^{1.5} + O(\epsilon^{2.5})$, $c \neq 0$. In our computation above, we see that $T(\epsilon) = 2\pi - 4\sqrt{2}\epsilon^{1.5} + O(\epsilon^{2.5})$. Then, the precision of our solution $u(\cdot)$ on the first order is valid on smaller interval $[0, \frac{a}{\sqrt{\epsilon}}]$. So, the period T_ϵ is of order $\epsilon^{-\frac{1}{2}}$. This is a new result for the Lindstedt-Poincaré Method. The method can be applied for some less smooth functions but with a smaller time of validity.

3 Numerical Study of the Solutions

To solve this problem numerically in Scilab, two methods: Euler's and Runge-Kutta's methods will be used. The two methods converge differently to the exact solution. The differences can be seen clearly from the errors made by those methods. In this section, you will also see graphs in which the approximated and the exact solutions are plotted based on respective time and tables containing various values of numerical, exact solution and errors which can be a proof of the accuracy of these methods.

3.1 Case $\epsilon = 0$

3.1.1 Euler's Method

The general formula for Euler's method is

$$\begin{cases} \dot{y} & = f(t, x) \\ x(k+1) & = x(k) + hf(t, x(k)) \end{cases}$$

To solve the system (2) in previous section with Euler's method in Scilab, we need to transform the problem into the first order linear differential system with new variables.

With the problem

$$\begin{cases} u''(t) + u(t) = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$$

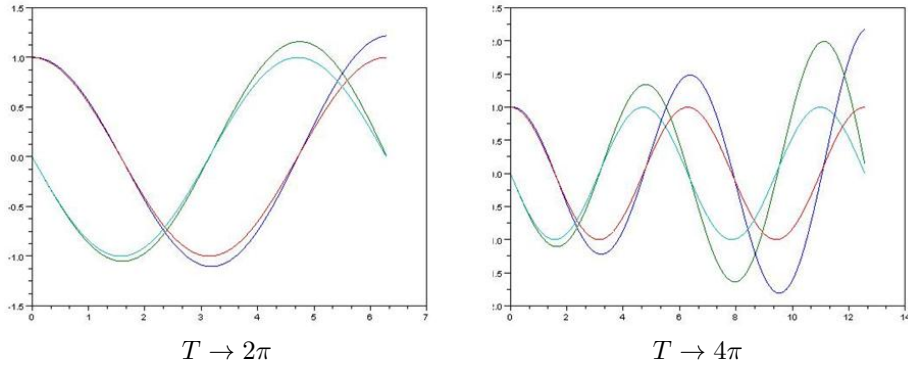
The new variables are defined as:

$$\begin{cases} u_1(t) = u(t) \\ u_2(t) = u'(t) \end{cases} \implies \begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = -u_1(t) \end{cases}$$

The problem in the first order differential equations is

$$\begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = -u_1(t) \\ u_1(0) = 1 \\ u_2(0) = 0 \end{cases}$$

In the above notations, the term $u_1(t)$ denotes the position of the mass and $u_2(t)$ denotes the velocity of the mass. In this case, the exact solution is $u(t) = \cos(t)$. The algorithm used to generate various values of numerical and exact solutions, graphs as well as the time T for this case can be found in the appendix A1. The following are graphs of both numerical solutions and the exact solutions in the case $\epsilon = 0$ when time evolves with $T \rightarrow 2\pi$ and $T \rightarrow 4\pi$.

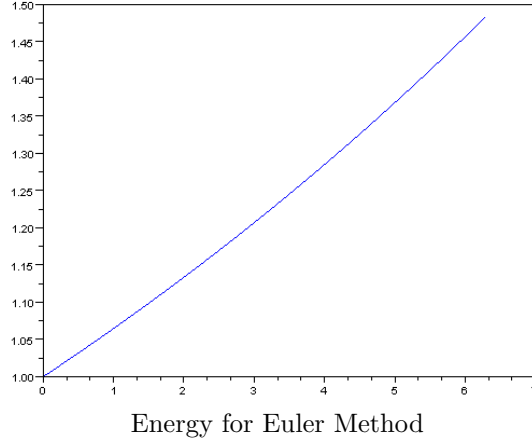


The following table contains the last five values of numerical and exact solutions at the respective time when $T \rightarrow 2\pi$. The notation $|E - N|$ is the error means the absolute value of the difference between the exact solution and the numerical solution.

T	Numerical Solution	Exact Solution	Error $ E - N $
6.0318579	1.1678164	0.9685832	0.1992332
6.0946897	1.1872769	0.9822873	0.2049896
6.0946897	1.1872769	0.9921147	0.1951622
6.2203535	1.2122898	0.9980267	0.2142631
6.2831853	1.2177068	1.	0.2177068

The maximum error in this case is 0.2177068. When $T \rightarrow 2\pi$, there is slightly different in the accuracy of the values of the exact solutions and the approximated solutions given by the Euler's method. The graphs of the approximated solutions and the exact solutions are almost coincident in this case.

However, the error increases when $T \rightarrow 4\pi$ that is $|E - N| = 1.1842021$. It is well known that Euler is not a good method to find periodic solution. Notice that the energy for Euler's method is not stable which show that the solution diverges.



3.1.2 Runge-Kutta's Method

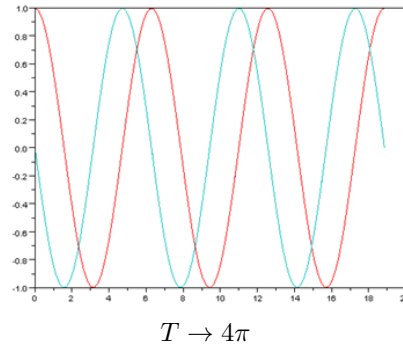
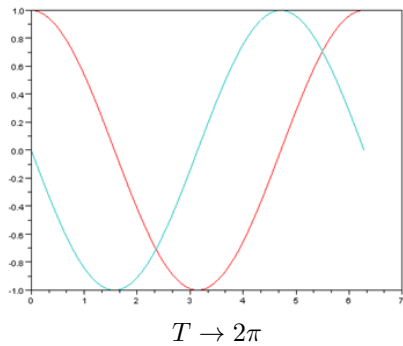
The general formula for Runge-Kutta's of order four is

$$\begin{cases} x(k+1) &= x(k) + hs \\ s &= \frac{1}{6}(s_1 + 2s_2 + 2s_3 + s_4) \end{cases}$$

where

$$\begin{cases} s_1 &= f(t(k), x(k)) \\ s_2 &= f(t(k) + \frac{h}{2}, x(k) + \frac{h}{2}s_1) \\ s_3 &= f(t(k) + \frac{h}{2}, x(k) + \frac{h}{2}s_2) \\ s_4 &= f(t(k) + h, x(k) + hs_3) \end{cases}$$

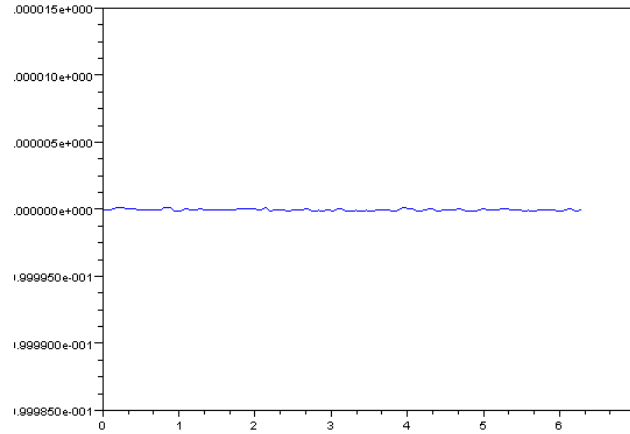
The algorithm in Runge-Kutta's method used to generate the numerical and exact solutions, graphs as well as time T for this case can be found in appendix A2. In this case, the exact solution is $u(t) = \cos(t)$. Here is the figure of numerical solution given by Runge-Kutta's plotted against the exact solution with two different time periods when $T \rightarrow 2\pi$ and $T \rightarrow 4\pi$.



The table below shows the last five values of numerical solutions, the exact solutions and the error $|E - N|$ with the respective time when $T \rightarrow 2\pi$.

T	Numerical Solutions	Exact Solutions	Error $ E - N $
6.0318579	0.9685829	0.9685832	0.0000003
6.0946897	0.9822871	0.9822873	0.0000002
6.0946897	0.9921146	0.9921147	0.0000001
6.2203535	0.9980266	0.9980267	0.0000001
6.2831853	1.	1.	0

From the table, the exact solutions and the numerical solutions are slightly different even time T increased and the error when $T \rightarrow 2\pi$ is 0. When $T \rightarrow 4\pi$, the error is estimated to be 0.0000027. So, the Runge-Kutta method works better for approximating the periodic solutions. Notice that the energy of the Runge-Kutta method is almost stable that is the graph of the energy is close to the horizontal line which shows that the numerical solution converges faster to the exact solution.



Energy Plot of Runge-Kutta's Method

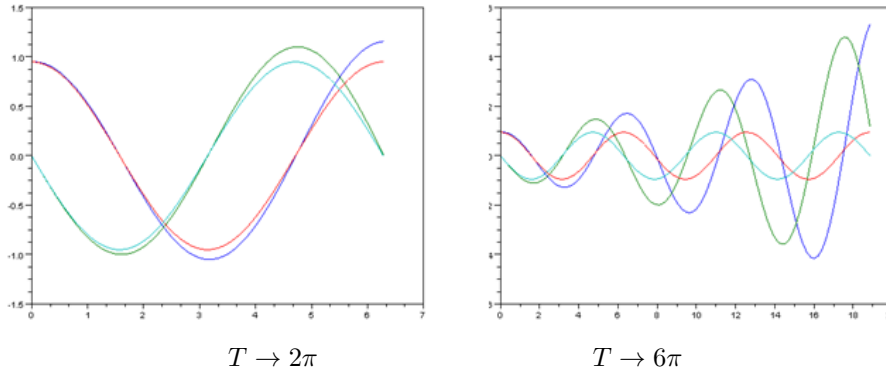
3.2 Case $-1 < \epsilon < 0$

3.2.1 Euler's Method

In this case, the problem in the first order derivative becomes

$$\begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = -u_1(t) \\ u_1(0) = 1 + \epsilon \\ u_2(0) = 0 \end{cases}$$

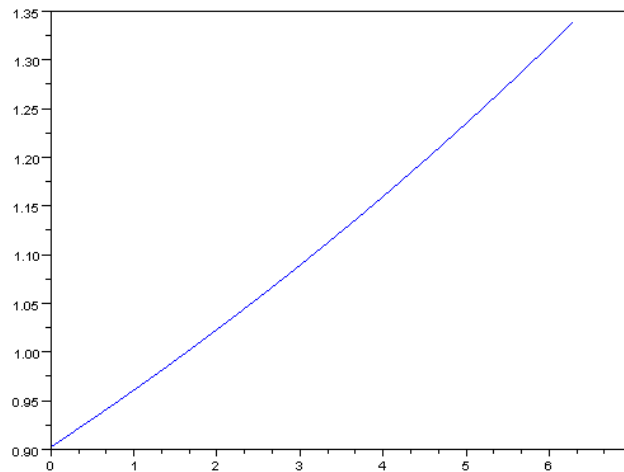
The algorithm with 100-step Euler's approximation at the value of $\epsilon = -0.05$ is used to generate the data and can be found in appendix A3. In the case $\epsilon < 0$, as found above, the exact solution is $u(t) = (1 + \epsilon) \cos(t)$. The following figure contains the graphs of numerical solutions plotted against the exact solutions with $\epsilon = -0.05$ when $T \rightarrow 2\pi$ and $T \rightarrow 6\pi$.



The table below shows the last five values of numerical solutions given by Euler's method, the exact solution, the error $|E - N|$ and the respective time T in the case $\epsilon < 0$. When $T \rightarrow 2\pi$, it shows that the maximum error made by the Euler's method is 0.2068215. When we increase the time to be 6π , the maximum error is significantly large that is $|E - N| = 4.3638501$. So, in this case also, the Euler method does not converge to the exact solution.

T	Numerical Solutions	Exact Solutions	Error $ E - N $
6.0318579	1.1094256	0.920154	0.1892716
6.0946897	1.127913	0.9331729	0.1947401
6.0946897	1.1420206	0.9425090	0.1995116
6.2203535	1.1516753	0.9481254	0.2035499
6.2831853	1.1568215	0.95	0.2068215

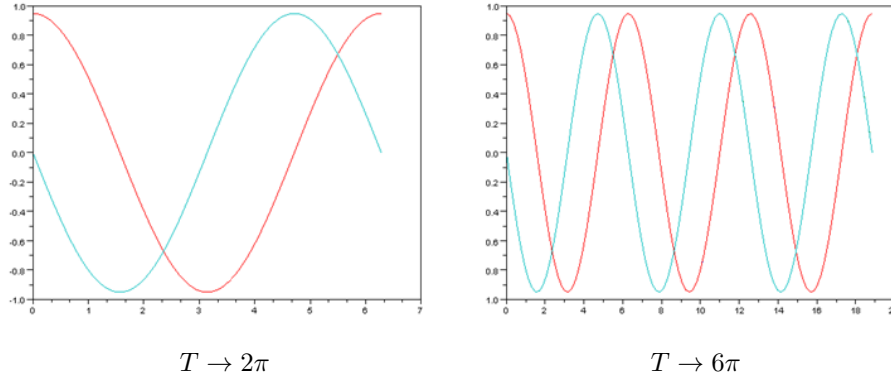
Notice that the Energy plot in this case is still unstable that is the graph of the energy is not close to the horizontal line.



Energy Plot of Euler with $\epsilon < 0$

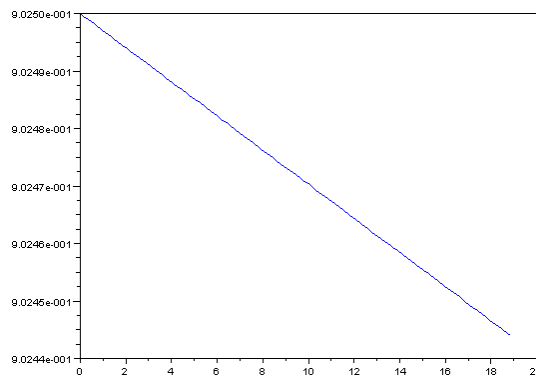
3.2.2 Runge-Kutta's Method

For Runge-Kutta's method, 100-step algorithms will also be used to generate the data and plot the graphs. The algorithms can be found in appendix A4. The following figure is the graph of numerical solution given by the Runge-Kutta's method plotted against the exact solution with $\epsilon = -0.05$ when $T \rightarrow 2\pi$ and $T \rightarrow 6\pi$.



The following table shows the last five values of numerical solutions given by Runge-Kutta's method and the exact solution and respective time T in the case $\epsilon < 0$ and when $T \rightarrow 6\pi$. It shows that the error made by the Runge-Kutta's method at time 6π is 0.0000295.

T	Numerical Solutions	Exact Solutions	Error $ E - N $
18.095574	0.6923773	0.6925202	0.0001429
18.284069	0.8019907	0.8021115	0.0001208
18.472565	0.8831937	0.8832877	0.000094
18.66106	0.9331097	0.9331729	0.0000632
18.849556	0.9499705	0.95	0.0000295



Energy Plot of the Runge-Kutta's method with $\epsilon = -0.05$

Notice that the graph of the energy is no longer stable in this case. The graph is not close to the horizontal line, so in the case of $\epsilon < 0$, both methods: Euler or Runge-Kutta can not give the exact solution to the problem.

3.3 Case $0 < \epsilon \leq 1$

From the first two cases, we notice that the Runge-Kutta's method works better than the Euler's method, so in the case $\epsilon > 0$, we will use only the Runge-Kutta's method for numerical computation. In section 2, we know that in the time interval $] -t_1, t_1[$, $u(t) > 1$, so the problem in new variables is written as the following:

$$\begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = -u_1(t) - \epsilon[u_1(t) - 1] \\ u_1(0) = 1 + \epsilon \\ u_2(0) = 0 \end{cases} = \begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = -(1 + \epsilon)u_1(t) + \epsilon \\ u_1(0) = 1 + \epsilon \\ u_2(0) = 0 \end{cases}$$

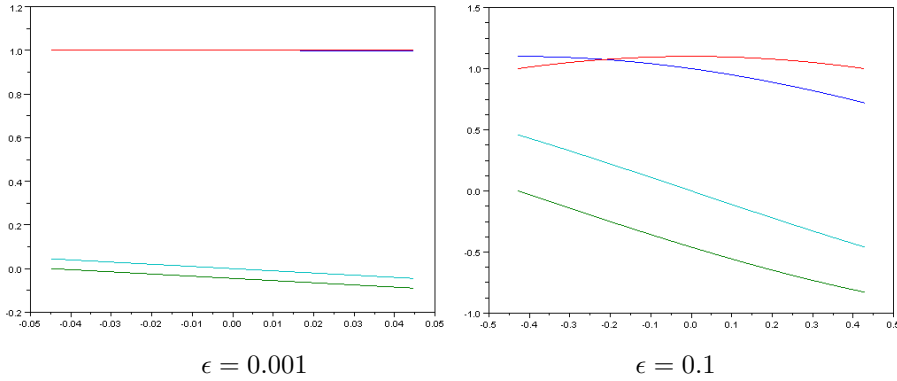
This new system will be used in numerical Scilab code to plot the graphs and generate different values of numerical and exact solutions. With this problem, the exact solution is $u(t) = \frac{\epsilon}{1+\epsilon} + \frac{\epsilon^2 + \epsilon + 1}{1+\epsilon} \cos(\sqrt{1+\epsilon}t)$ where $t_1 = \frac{\arccos(\frac{1}{1+\epsilon+\epsilon^2})}{\sqrt{1+\epsilon}}$. Notice that the problem contains the nonlinear term $u - 1$.

In the interval $]t_1, t_2[$, we have $u(t) < 1$ and in this case the problem written in a new variable form with a new initial time at $t = t_1$ is:

$$\begin{cases} u_1'(t) = u_2(t) \\ u_2'(t) = u_1(t) \\ u_1(t_1) = 1 \\ u_2(t_1) = -\sqrt{\epsilon(\epsilon^2 + \epsilon + 2)} \end{cases}$$

The exact solution for this case is $u(t) = \cos(t - t_1) - \sqrt{\epsilon(\epsilon^2 + \epsilon + 2)} \sin(t - t_1)$. Unlike the previous case, the problem now contains no nonlinear term $u - 1$, so it is a linear problem with new initial data. In this case, you will see the graphs of numerical and exact solutions are plotted and values of numerical and exact solution will also be generated with different values of ϵ . The Scilab algorithms used for the interval $] -t_1, t_1[$ can be found in appendix A5 while the algorithms for interval $]t_1, t_2[$ can be found in appendix A6.

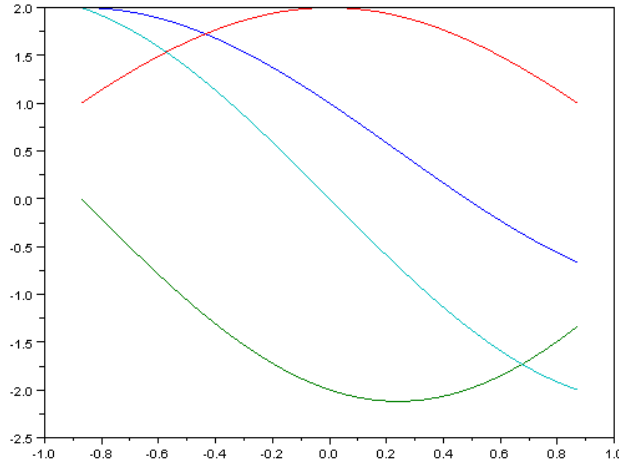
Here are graphs of numerical solution plotted against exact solution with two different values of ϵ in the interval $] -t_1, t_1[$. Notice how the notion of the graphs change with the value of ϵ . With $\epsilon = 0.001$, the graphs of numerical and exact solution are almost coincident, but with $\epsilon = 0.1$, the graphs get away from each other.



The following table shows the last five values of numerical solution and exact solution with $\epsilon = 0.001$ and the last five values of numerical and exact solutions with $\epsilon = 0.1$ with respective time $T \in]-t_1, t_1[$. Notice that the errors in this case are significantly small, especially with $\epsilon = 0.001$. We see that the Runge-Kutta's method works best for small value of ϵ in this case.

T	Numerical Solutions	Exact Solutions	Error $ E - N $
0.41126502	0.9973153	1.0001536	0.0028383
0.42020556	0.9972381	1.0001164	0.0028783
0.4291461	0.9971602	1.0000784	0.0029182
0.43808665	0.9970815	1.0000396	0.0029581
0.44702719	0.997002	1.	0.002998
0.3938174	0.7477511	1.0151407	0.2673896
0.4023786	0.7408463	1.0114665	0.2706202
0.4109398	0.7338891	1.0077182	0.2738291
0.4195011	0.7268801	1.0038959	0.2770158
0.4280623	0.7198198	1.	0.2801802

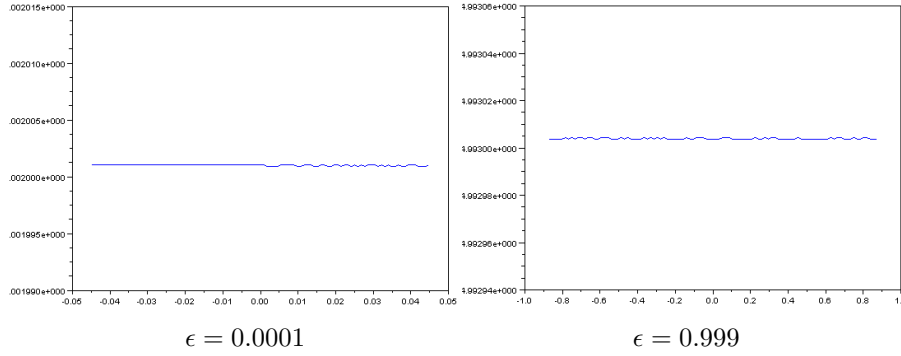
Now see the graphs of numerical and exact solutions when ϵ get larger. In this case, the graph of numerical solution leaves the graph of exact solution which produces significantly large errors. The figure is plotted against $\epsilon = 0.999$.



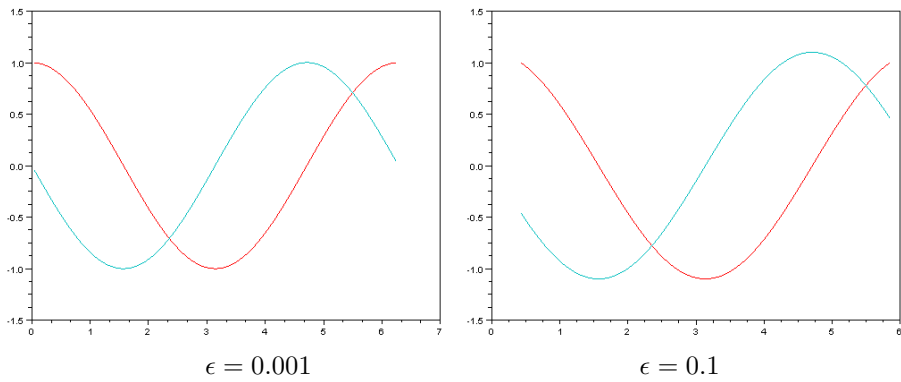
The following table shows the last five values of numerical and exact solutions with $\epsilon = 0.999$ with respective time $T \in]-t_1, t_1[$. Notice that the errors in this case are significantly large. So, the Runge-Kutta's method fails to provide accurate computation for nonlinear problem, especially when ϵ become larger.

T	Numerical Solutions	Exact Solutions	Error $ E - N $
0.8004730	- 0.5583105	1.135354	1.6936645
0.8178746	- 0.5838862	1.1020416	1.6859278
0.8352762	- 0.6088104	1.0683649	1.6771753
0.8526777	- 0.6330679	1.0343442	1.6674121
0.8700793	- 0.6566442	1.	1.6566442

Now let's see the energy plot below with two values of ϵ . The plot shows that the energy is approximately close to the horizontal line with the two values of ϵ , but the previous case seems better.



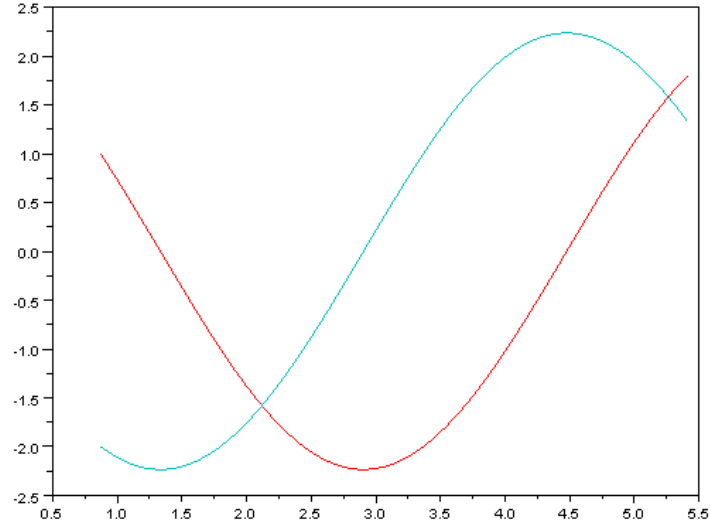
Next, see the graphs in the case $u(t) < 1$ where $t \in]t_1, t_2[$. Notice the values of numerical and exact solutions get very close to each other. The error in this case is very small which implies that the method provides accurate solution.



The table below contains the last five values of numerical and exact solutions with $\epsilon = 0.001$ and the last five values with $\epsilon = 0.1$ with respective time $T \in]t_1, t_2[$. The error in this case is approximated to be zero.

T	Numerical Solution	Exact Solution	Error $ E - N $
5.9907314	0.9584966	0.9584968	0.0000002
6.0526692	0.9745220	0.9745222	0.0000002
6.114607	0.9868100	0.9868101	0.0000001
6.1765448	0.9953135	0.9953136	0.0000001
6.2384826	0.9999999	1.	0.0000001
5.6380405	0.8809532	0.8809535	0.0000003
5.6923111	0.9154293	0.9154295	0.0000002
5.7465818	0.9472099	0.9472101	0.0000002
5.8008524	0.9762013	0.9762015	0.0000002
5.855123	1.0023182	1.0023184	0.0000002

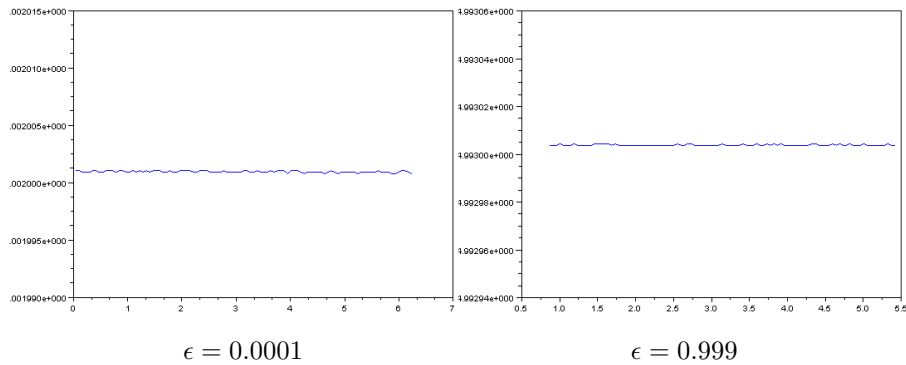
Now see the graphs of numerical and exact solutions when ϵ get larger. In this case, the error are still significantly small. The graphs are plotted against $\epsilon = 0.999$.



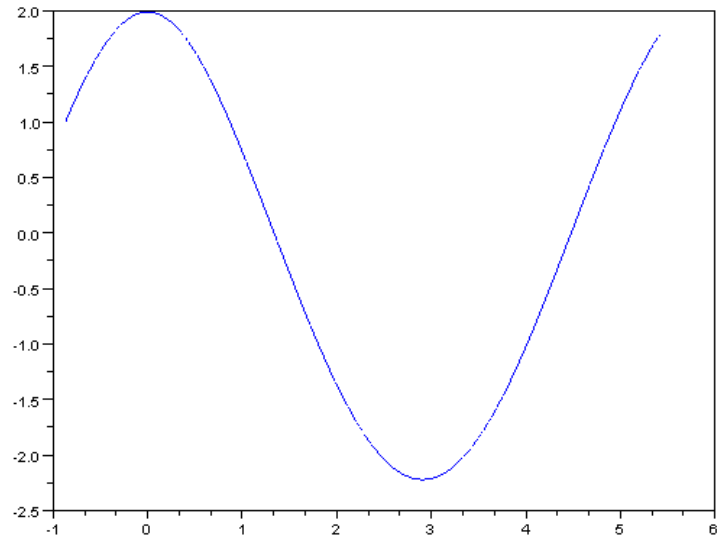
The following table shows the last five values of numerical and exact solutions with $\epsilon = 0.999$ along with respective time $T \in]t_1, t_2[$. Notice that the errors in this case are still significantly small, especially when $T \rightarrow t_2$. The error is approximated to be zero in this case. So, the Runge-Kutta's method gives very accurate solutions to the linear problem even ϵ become larger.

T	Numerical Solutions	Exact Solutions	Error $ E - N $
5.2311017	1.5314812	1.5314815	0.0000003
5.2765258	1.6037872	1.6037875	0.0000003
5.3219499	1.6727847	1.6727849	0.0000002
5.367374	1.7383311	1.7383314	0.0000003
5.4127981	1.8002915	1.8002917	0.0000002

Again notice that the graph of the energy with $\epsilon = 0.001$ and $\epsilon = 0.999$. The graphs however are approximately close to the horizontal line.



The graph of the exact solution when $u(t) > 1$ and $u(t) < 1$ in the interval $] - t_1, t_1[$ and $]t_1, t_2[$ below is a continuous map. See the following figure.



4 Appendix: *Numerical Algorithms*

Appendix-A1

```
// Function used to solve the ODE by Euler Method
// Case epsilon=0 with period 2*pi

function [T,U]=euler(f,t0,u,t1,n)
    h=(t1-t0)/n;
    T=t0;
    U=u';
    for i=1:n
        t0=t0+h;
        u=u+h*f(t0,u);
        T=[T;t0];
        U=[U;u'];
    end
endfunction

// Numerical solution of the problem is given by
function nu=f(t0,u)
    nu=u;
    nu(1)=u(2);
    nu(2)=-u(1);
endfunction
[T,U]=euler(f,0,[1;0],2*pi,100)
plot(T,[U,cos(T),-sin(T)])
```

Appendix-A2

```
//Runge Kutta method with epsilon=0

function [T,U]=rk4(f,t,u,t1,n)
    h=(t1-t)/n;
    T=t;
    U=u';
    for i=1:n
        k1=f(t,u);
        k2=f(t+h/2,u+h*k1/2);
        k3=f(t+h/2,u+h*k2/2);
        k4=f(t+h,u+h*k3);
        k=(k1+2*k2+2*k3+k4)/6;
        t=t+h;
        u=u+h*k;
    T=[T;t];
    U=[U;u'];
    end
endfunction

// Numerical solution of the problem is given by
function up=f(t,u)
```

```

        up=u;
        up(1)=u(2);
        up(2)=-u(1);
endfunction
    [T,U]=rkn(f,0,[1;0],20*%pi,100)
    plot(T,[U,cos(T),-sin(T)])

```

Appendix-A3

```

// Function used to solve the ODE by Euler Method
// Case epsilon=-0.05
function [T,U]=euler(f,t0,u,t1,n)
    h=(t1-t0)/n;
    T=t0;
    U=u';
    for i=1:n
        t0=t0+h;
        u=u+h*f(t0,u);
        T=[T;t0];
        U=[U;u'];
    end
endfunction

// Numerical solution of the problem is given by
function nu=f(t0,u)
    nu=u;
    nu(1)=u(2);
    nu(2)=-u(1);
endfunction
    [T,U]=euler(f,0,[1-0.05;0],2*%pi,100)
    plot(T,[U,(1-0.05)*cos(T),-(1-0.05)*sin(T)])

```

Appendix-A4

```

// The algorithm in Runge-Kutta method
// Epsilon=-0.05
function [T,U]=rkn(f,t,u,t1,n)
    h=(t1-t)/n;
    T=t;
    U=u';
    for i=1:n
        k1=f(t,u);
        k2=f(t+h/2,u+h*k1/2);
        k3=f(t+h/2,u+h*k2/2);
        k4=f(t+h,u+h*k3);
        k=(k1+2*k2+2*k3+k4)/6;
        t=t+h;
        u=u+h*k;
        T=[T;t];
        U=[U;u'];
    end
end

```

```

endfunction
// Numerical solution of our problem is given by
function up=f(t,u)
    up=u;
    up(1)=u(2);
    up(2)=-u(1);
endfunction
[T,U]=rkn(f,0,[1-0.05;0],2*%pi,100)
plot(T,[U,(1-0.05)*cos(T),-(1-0.05)*sin(T)])

```

Appendix-A5

```

// Runge-Kutta method used to approximate the solution
of differential equation for u(.)>1
//Epsilon=0.001
function [T,U]=rkn(f,t,u,t1,n)
    h=(t1-t)/n;
    T=t;
    U=u';
    for i=1:n
        k1=f(t,u);
        k2=f(t+h/2,u+h*k1/2);
        k3=f(t+h/2,u+h*k2/2);
        k4=f(t+h,u+h*k3);
        k=(k1+2*k2+2*k3+k4)/6;
        t=t+h;
        u=u+h*k;
        T=[T;t];
        U=[U;u'];
    end
endfunction
// Numerical solution is given by:
function up=f(t,u)
    e=0.001;
    up=u;
    up(1)=u(2);
    up(2)=-(1+e)*u(1)+e;
endfunction
e=0.001;
ta=acos(1/(1+e+e^2))/sqrt(1+e);
[T,U]=rkn(f,-ta,[1+e;0],ta,100)
plot(T,[U,e/(1+e)+((1+e+e^2)/(1+e))*cos(sqrt(1+e)*T),
-sqrt(1+e)*((1+e+e^2)/(1+e))*sin(sqrt(1+e)*T)])

```

Appendix-A6

```

// Runge-Kutta method used to approximate the solution
of differential equation for u(.)<1
//Epsilon=0.001
function [T,U]=rkn(f,t,u,t1,n)
    h=(t1-t)/n;

```

```

        T=t;
        U=u';
    for i=1:n
        k1=f(t,u);
        k2=f(t+h/2,u+h*k1/2);
        k3=f(t+h/2,u+h*k2/2);
        k4=f(t+h,u+h*k3);
        k=(k1+2*k2+2*k3+k4)/6;
        t=t+h;
        u=u+h*k;
        T=[T;t];
        U=[U;u'];
    end
endfunction
// Numerical solution is given by:
function up=f(t,u)
    e=0.001;
    up=u;
    up(1)=u(2);
    up(2)=-u(1);
endfunction
e=0.001;
ta=acos(1/(1+e+e^2))/sqrt(1+e);
[T,U]=rk4(f,ta,[1;-sqrt(e*(e^2+e+2))],2*pi-ta,100)
plot(T,[U,cos(T-ta)-sqrt(e*(e^2+e+2))*sin(T-ta),
-sin(T-ta)-sqrt(e*(e^2+e+2))*cos(T-ta)])

```

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